

## The spectral signature of salt fingers

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**Abstract**—The spectra of either temperature, salinity or vertical velocity which might be measured by a sensor moving through a field of salt fingers are considered. A theoretical calculation is presented which attempts to incorporate the deviations from squareness and perfect orientation of the salt fingers. The resulting spectra rise to a peak close to  $\omega = (\sqrt{2})kv$  and decrease rapidly for larger  $\omega$ , where  $\omega$  is the temporal frequency of the spectrum,  $k$  the dominant wave number of the salt fingers and  $v$  the horizontal velocity of the sensor. A spectrum of vertical velocity of laboratory-generated salt fingers is shown to be in close agreement with the theoretical prediction.

OVER THE last twenty years the concept of salt fingers has evolved from being 'An oceanographical curiosity' (STOMMEL, ARONS and BLANCHARD, 1956) to being an observed fact (WILLIAMS, 1974, 1975). By using a collimated laser beam, Williams' instrument obtained a series of photographs 5 cm in diameter of salt fingers in the Mediterranean outflow which were very similar to those easily observed in a laboratory. While an instrument which allows one to actually 'see' phenomena occurring at a depth of over 1000 m is a very valuable new tool, it is not feasible to use it to obtain a quantitative estimate of what proportion of a (large) section of ocean is salt-fingering at any particular time. This estimate is important because it will differentiate between the notion that salt fingers occur in a significant fraction of the ocean, as conjectured by HUPPERT and MANINS (1973), and the notion that they occur in only very isolated spots.

Recently, efforts have been made to use a small temperature and salinity sensor which could be towed behind a ship to yield data which might be analysed to determine over what fraction of the path the sensor was moving through a salt-finger field. But how will the record appear when the sensor is in a salt-finger field; that is, what is an observable signature of salt fingers? The examination of this question is the aim of the present note.

A careful investigation of the raw line trace, output by the sensor, would be a very time-consuming and not very profitable task. The best means of identification of salt fingers would seem to come from the spectrum, and we concentrate on this here. We calculate only the spectrum due to salt fingers; those portions of the record not due to salt fingers will contribute to the background spectrum and in practice this influence could be removed either before or after calculation of the spectrum.

The simplest representation of a salt-finger field reflects the fact that individual salt fingers are very much longer than they are thick and have a nearly square plan form (SHIRTCLIFFE and TURNER, 1970). Thus all variables are of the form

$$\phi = \phi_0 \sin kx \sin ky, \quad (1)$$

where  $\phi$  represents temperature, salinity or vertical velocity of amplitude  $\phi_0$ ,  $k$  is a wave number related to the large-scale temperature and salinity gradients and  $x$  and  $y$  are orthogonal, horizontal axes. The form (1) has been shown by HUPPERT and MANINS (1973) to be the only one representing a steady-state, square salt-finger field with purely vertical motion.

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Suppose that a variable of the form (1) is measured by a sensor moving horizontally with constant speed  $v$  along the straight line  $y = mx + c$ , where  $0 \leq m < \infty$ . Then by straightforward manipulations, of the sort described for example in JENKINS and WATTS (1968), it is possible to show that the temporal spectrum of the records  $F(\omega; m)$ , is given by

$$F(\omega; m) = \frac{1}{8}\pi\phi_0^2[\delta(\omega + \omega_+) + \delta(\omega - \omega_+) + \delta(\omega + \omega_-) + \delta(\omega - \omega_-)], \quad (2)$$

where

$$\omega_{\pm} = \frac{(m \pm 1)kv}{\sqrt{(m^2 + 1)}}, \quad (3a, b)$$

unless  $m = 0$  (propagation parallel to the  $x$  axis)  $m = 1$  (propagation parallel to a diagonal of the cell) or  $m = \infty$  (propagation parallel to the  $y$  axis). The results for these special cases are

$$F(\omega, 0) = \frac{1}{2}\pi\phi_0^2 \sin^2(kc)[\delta(\omega + kv) + \delta(\omega - kv)], \quad (4a)$$

$$F(\omega, 1) = \frac{1}{8}\pi\phi_0^2 \{2(1 + \cos 2kc)\delta(0) + \delta[\omega + (\sqrt{2})kv] + \delta[\omega - (\sqrt{2})kv]\} \quad (4b)$$

and

$$F(\omega, \infty) = \frac{1}{2}\pi\phi_0^2 \sin^2(kx_{\infty}) \times [\delta(\omega + kv) + \delta(\omega - kv)], \quad (4c)$$

where in order to derive (4c) the sensor is assumed to move along  $x = x_{\infty}$ . Note that generally  $F(\omega; m)$  is independent of  $c$  because  $c$  acts only as a phase which does not appear in the spectrum;  $m = 0, 1, \infty$  are special cases for which the sensor moves through a particular geometry.

The most surprising and important feature of the spectrum is that the contributions, or delta functions, are only for  $|\omega| \leq (\sqrt{2})kv$  or for  $|\alpha| \leq (\sqrt{2})k$  where  $\alpha$  is the spatial frequency. Thus the spectrum can be non-zero only for sufficiently large wavelengths or periods. The possibly intuitive statement that the sensor 'clips' the corner of cells and hence there is a significant contribution by small wavelengths is incorrect. The measured quantity tends to zero at the cell boundaries and the sensor measures not the small-wavelength corners but the wavelengths intrinsic

to (1). The minimum wavelength over all orientations thus corresponds to the distance between the centres of two diagonally aligned cells, that is  $|\alpha| \leq (\sqrt{2})k$ .

The derivation of the spectral form (2) omits two important, not entirely independent features. First, usually the total record from the sensor will not come from salt fingers all aligned with the  $x$  and  $y$  axis in the form (1). Rather, different 'clumps' of fingers will be differently oriented. The effect of this on the spectrum will be the same as if the sensor passed many times through a salt-finger field of the form (1) making an angle  $\theta \equiv \tan^{-1} m$  with the  $x$  axis, where  $\theta$  is a random variable uniformly distributed over  $[0, \frac{1}{2}\pi]$ . The spectrum so measured,  $\mathcal{F}(\omega)$ , is given by

$$\mathcal{F}(\omega) = \frac{2}{\pi} \int_0^{\pi/2} F(\omega; m) d\theta, \quad (5)$$

$$= \frac{1}{2}\phi_0^2(2k^2v^2 - \omega^2)^{-\frac{1}{2}} \quad [\omega < (\sqrt{2})kv] \quad (6a)$$

$$= 0 \quad [\omega > (\sqrt{2})kv], \quad (6b)$$

and this simple function is graphed in Fig. 1. Note the rise in the spectrum to an (infinite) peak at  $\omega = (\sqrt{2})kv$  and its zero value for  $\omega > (\sqrt{2})kv$ .

The second feature omitted from the derivation of (2) is that laboratory experiments indicate that neighbouring salt fingers are not perfectly aligned with respect to each other. Instead, over a linear distance of about five fingers the orientation with respect to fixed axes varies significantly. This leads to the concept of five or so fingers aligned in a 'salt-hand' and many hands, randomly aligned. It seems impossible at the present to quantify deductively such behaviour, but many partially *ad hoc* relationships can be written down. One of the simplest, which retains the observed squareness of each cell and is suitably symmetric is

$$\phi = \phi_0 \sin \{k[x - \eta \sin(l\psi_1)]\} \times \sin \{k[y - \eta \sin(l\psi_2)]\}, \quad (7)$$

where  $\psi_1$  and  $\psi_2$  are independent random variables uniformly distributed over  $[0, 2\pi]$  and for the moment  $l$  and  $\eta$  will be considered constants, though strictly they too are random variables. After a number of pages of manipulations,

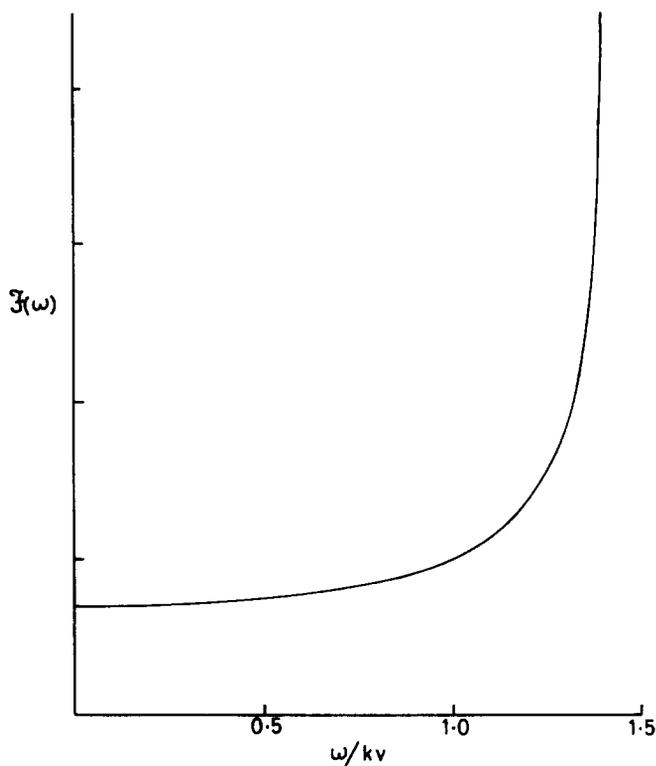


Fig. 1. The spectrum  $\mathcal{F}(\omega)$  obtained after many passes with randomly distributed orientations through a square salt-finger field of the form given by  $\phi = \phi_0 \sin kx \sin ky$ .

with much reference to the standard works of classical analysis, we obtain

$$\mathcal{F}(\omega) = \frac{1}{4} \phi_0^2 \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \frac{J_r^2(\eta k) J_s^2(\eta k)}{\sqrt{\{(rl+k)^2 + (sl+k)^2\} v^2 - \omega^2}}, \tag{8}$$

where  $J_n(x)$  is a Bessel function. This form is still somewhat unsatisfactory due to the singularities at the zeros of the denominator, which are removed by considering  $l$  to be a random variable. The probability distribution

$$\Phi(l) = (0.7k)^{-1} \times \begin{cases} 1 + 5l/k & (0 \leq l/k \leq \frac{1}{2}) \tag{9a} \\ \frac{2}{5}(l/k)^{-2} & (l/k \geq \frac{1}{2}) \tag{9b} \end{cases}$$

models the loss of orientation over approximately five fingers and is of a form which allows the  $l$ -integration to be performed analytically. With

this distribution and  $\eta k = 1$  numerical evaluation of the spectrum leads to the curve presented in Fig. 2. Note the significant amount of power for small  $\omega$ , the rise to a peak close to  $\omega = (\sqrt{2})kv$  and the rapid falloff thereafter. The form of the spectrum at low frequencies is quite sensitive to the exact specification of  $\Phi(l)$ . For frequencies close to and above  $(\sqrt{2})kv$ , however, reasonable probability distributions for  $l$  other than (9) yield similar spectra as do further calculations treating  $\eta$  as a random variable.

If the sensor is moving at an angle to the horizontal, the above results are valid if  $v$  is regarded as the horizontal speed of the sensor. Should the sensor move in random directions through drawn-out salt sheets, rather than salt fingers, the expressions for  $\mathcal{F}(\omega)$  are unaffected.

In order to test these ideas, spectra of the vertical velocity of laboratory generated salt fingers were obtained. Salt fingers were produced in an

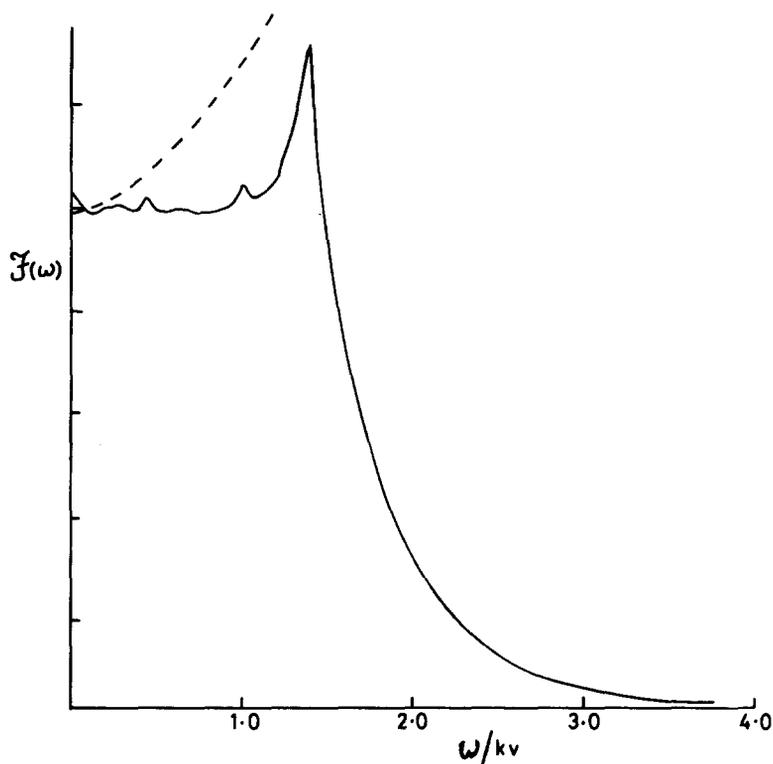


Fig. 2. The spectrum  $\mathcal{F}(\omega)$  obtained after many passes with randomly distributed orientations through a salt-finger field of the form given by (7) with  $l$  randomly distributed as indicated in (9) and  $\eta k = 1$  is the solid curve. The spectrum  $\mathcal{F}(\omega)$  of Fig. 1 is the dashed curve.

interface between two well-mixed layers and the vertical velocity measured by observing the distortion of a horizontal line of dye. The details of the experiment have been reported elsewhere (LINDEN, 1973) and the reader is referred to the original paper for further information. A dye line spanned about 50 fingers and the photographs of the distorted trace [see LINDEN (1973), Fig. 3 for an example] were enlarged and digitized at 256 equally spaced points. Spectra were calculated from this digital information. In every case the fingers were at least 5 cm long and only the dye line at the centre of the interface was used. This was done to avoid end-effects with the intention that the vertical velocity field could be reasonably well described by (7).

An example of one of these spectra is shown on Fig. 3. Considerable variation in the spectra were found at low wave numbers, and the example

chosen is the one which most closely resembles the spectrum shown on Fig. 2. At higher wave numbers the spectra were found to be quite similar. The main features to note are the peak in the spectrum at  $\alpha = \alpha_1$ , and the rapid drop in the spectrum at wave numbers greater than  $\alpha_1$ . Both of these features are predicted by (6) and were present in all the other spectra calculated from the dye traces.

LINDEN (1973) has shown experimentally that the width of a salt finger  $L$  is related to the vertical temperature gradient  $T_z$  along the finger by a relationship of the form

$$L = cT_z^{-1/2}, \quad (10)$$

where  $c$  is a constant whose value is dependent upon the molecular properties of the fluid. With  $L$  in cm and  $T_z$  in  $^{\circ}\text{C cm}^{-1}$  the laboratory measurements were best represented by  $c = 0.23$ .

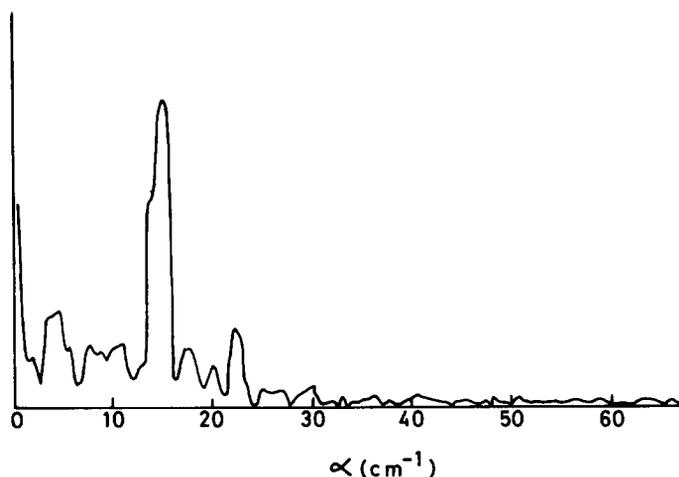


Fig. 3. The spectrum of the vertical velocity of salt fingers generated in the laboratory. The fingers were produced at an interface between a layer of warm water with high saccharinity overlying a layer of cooler, fresher water. The mean vertical temperature gradient through the interface was  $0.52^{\circ}\text{C cm}^{-1}$ . The plot shows the squared sum of the transform of the velocity (in arbitrary units) plotted against the horizontal wave number.

Equation (10) was originally obtained by STERN (1960) on the basis of calculating the most rapidly growing mode in a linear stability analysis and the experiments show that it also applies to finite amplitude salt fingers. WILLIAMS (1974) observed salt fingers beneath the Mediterranean outflow and found that (10) predicted the width of these fingers also. Applying (10) to the case shown on Fig. 3, with  $T_z = 0.52^{\circ}\text{C cm}^{-1}$ , gives  $L = 0.27\text{ cm}$ . According to (6) the peak of the spectrum should occur at  $\alpha = (\sqrt{2})k = \pi(\sqrt{2})L^{-1} = 16.5\text{ cm}^{-1}$ . This is slightly larger than the observed peak at  $\alpha = 14.7\text{ cm}^{-1}$  but given the uncertainties in the experimental determination of  $L$  from (10) we consider this agreement to be quite satisfactory.

In view of the somewhat *ad hoc* nature of the statistical description of the salt fingers, the agreement between the theoretical and observed spectra is very encouraging. We should add the modifying remark, however, that one feature of the theoretical derivation which does not pertain to the laboratory experiments is the assumption that  $m$  is randomly distributed [see equation (5)]. In each of the experimental spectra the value of  $m$  must be regarded as fixed, although unknown. But this does not change the essential feature of

the spectrum, namely, the peak at  $\alpha = (\sqrt{2})k$  and the rapid decay of the spectrum for  $\alpha > (\sqrt{2})k$ .

Recently, MAGNELL (1976) has reported measurements of horizontal conductivity fluctuations measured on interfaces in the region of the Mediterranean outflow. The data indicate the presence of small-scale structure on these interfaces which, as the vertical temperature and salinity gradients are in the appropriate sense, may be salt fingers. Magnell presents a spectrum of these conductivity fluctuations measured at a single interface. The agreement between this one spectrum and that described by (8) is not good, but it is not clear how typical Magnell's spectrum is. The acquisition and analysis of more oceanographic data to compare with the present model would be useful. In particular, the use of a sensor together with an optical instrument such as that of WILLIAMS (1974) could provide *in situ* spectra to be compared with the present model as well as supplying independent evidence of the presence of salt fingers.

In conclusion, we suggest that salt fingers might be detected by analysing the spectra of velocity, temperature or salinity obtained from a sensor moving through the fingers. Our theoretical

model and laboratory experiment predict the low frequency portion of the spectrum will contain almost no discernible signature, except for the rise to a peak at  $\omega = (\sqrt{2})kv$  or  $\alpha = (\sqrt{2})k$  and that for higher frequencies the spectrum will decrease very rapidly.

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